ON THE MINIMIZING OF A CLASS OF DEFINITE INTEGRALS.*

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Introduction. Several special problems in the calculus of variations lead to the consideration of a definite integral of the form

$$(1) \qquad \int_{x_0}^{x_1} \frac{[1+y'^2(x)]^m}{y''(x)}\, dx.$$

For example, Euler's historic problem of finding the curve which with its evolute encloses a minimum area† gives rise to the particular case $m = 2$ in the above integral. The case $m = 1$ arises in obtaining the curve which with its caustic encloses a minimum area.‡ In finding the curves of minimum mean radius of curvature with respect to the arc and with respect to the abscissa, we are led to the cases $m = 2$ and $m = \frac{3}{2}$ respectively. It would thus seem desirable to develop a theory for the minimizing of the integral (1), to which, moreover, peculiar interest attaches on account of the fact that the second derivative appears in the integrand, comparatively few problems of that kind having been completely solved.

It is the purpose of this paper to derive such a theory. In section 1 are obtained the equations of the extremals, or curves that minimize the integral (1). In section 2 it is shown that these curves actually do furnish a minimum value for the integral. The question of the determination of the arbitrary constants that occur in the solution of section 1 is studied in section 3. Sections 4—6 treat various cases of variable end conditions. Finally, in section 7, a concrete illustration is given by means of the curve of minimum mean radius of curvature with respect to x.

1. **The minimizing curves.** Our problem is that of finding a curve $y = y(x)$ which joins the point $P_0\ (x_0, y_0)$ to the point $P_1\ (x_1, y_1)$ $(x_0 < x_1)$, and which gives to the integral (1) a smaller value than any neighboring curve. Let us assume that such a curve exists, and consider comparison curves of

* Presented to the American Mathematical Society, April 14, 1922.

† See Todhunter, Researches in the calculus of variations, chapter XIII.

‡ See Rider, The minimum area between a curve and its caustic, Bull. Amer. Math. Soc., vol. 27, p. 279; also Dunkel, The curve which with its caustic encloses the minimum area, Washington Univ. Studies, vol. 8, scientific series, p. 183.

the form $y = y(x) + \epsilon\eta(x)$. The value of the integral (1) for one of these comparison curves is

$$I(\epsilon) = \int_{x_0}^{x_1} \frac{[1 + (y' + \epsilon\eta')^2]^m}{y'' + \epsilon\eta''}\, dx.$$

Since the comparison curves reduce to the minimizing curves for $\epsilon = 0$, we must have $I'(0) = 0$ for a minimum. We find that

$$I'(0) = \int_{x_0}^{x_1} \left[\frac{2my'(1+y'^2)^{m-1}}{y''}\eta' - \frac{(1+y'^2)^m}{y''^2}\eta''\right] dx, \tag{2}$$

which, by the usual integration by parts employed in the calculus of variations, becomes

$$I'(0) = \left[\frac{2my'(1+y'^2)^{m-1}}{y''} + \frac{d}{dx}\frac{(1+y'^2)^m}{y''^2}\right]\eta\,\Bigg|_{x_0}^{x_1} - \frac{(1+y'^2)^m}{y''^2}\eta'\,\Bigg|_{x_0}^{x_1} \tag{3}$$

$$-\int_{x_0}^{x_1}\left[\frac{d}{dx}\frac{2my'(1+y'^2)^{m-1}}{y''} + \frac{d^2}{dx^2}\frac{(1+y'^2)^m}{y''^2}\right]\eta\, dx.$$

If it is prescribed that the end points P_0 and P_1 are fixed and that the minimizing curve shall have the slopes p_0 and p_1 at P_0 and P_1 respectively, then we choose as comparison curves those for which η and η' are zero at the end points, and therefore the first two expressions on the right side of equation (3) vanish. If the integral in (3) is also to vanish, it follows from the fundamental lemma of the calculus of variations that the coefficient of η in the integrand must be equal to zero. This gives a differential equation whose first integral is

$$\frac{2mp(1+p^2)^{m-1}}{q} + \frac{d}{dx}\frac{(1+p^2)^m}{q^2} = \text{constant} = 2A,$$

where we have set $p = y'$, $q = y''$. This reduces to

$$\frac{2mp}{1+p^2} - \frac{q'}{q^2} = \frac{Aq}{(1+p^2)^m}.$$

Multiplying by $dp = q\,dx$, we get

$$\frac{2mp\,dp}{1+p^2} - \frac{dq}{q} = \frac{Aq\,dp}{(1+p^2)^m},$$

or

$$d\log f = \frac{A\,dp}{f}, \tag{4}$$

where

$$f = \frac{(1+p^2)^m}{q}.$$

Integrating (4), replacing f by $(1+p^2)^m/q$, and solving for q, we obtain

$$q = \frac{(1+p^2)^m}{Ap+B}. \tag{5}$$

Since $q = dp/dx$, and $dy = p\,dx$, we find that

$$\begin{cases} x = \displaystyle\int_0^p \frac{Ap+B}{(1+p^2)^m}\,dp + C, \\ y = \displaystyle\int_0^p \frac{Ap^2+Bp}{(1+p^2)^m}\,dp + D, \end{cases} \tag{6}$$

in which C and D are arbitrary constants. Thus

The extremals are given by the parametric equations (6), *the parameter* p *being the slope of the curve at the point* (x, y).

It is not difficult to show that changing the sign of A merely reflects a given curve in the line $x = C$, and that changing the sign of B reflects it in the line $y = D$. Consequently, if $A = 0$ the curve is symmetric with respect to the line $x = C$, and if $B = 0$ it is symmetric with respect to the line $y = D$.

As a typical case we select $A < 0 < B$. Suppose $B = -rA$, $r > 0$. Then q is positive and the curve is concave upward as p increases from $-\infty$ to r, for which value q becomes infinite and the curve has a cusp. For $r < p < \infty$, q is negative and the curve is concave downward. By a study of dx/dp we discover that as p increases from $-\infty$ to r, x increases up to a maximum value at the cusp, and then recedes as p increases from r to ∞. By a study of dy/dp we learn that as p increases from $-\infty$ to 0, y decreases to a minimum value, and then as p increases to r, y increases to a maximum value at the cusp, and then decreases as p increases from r to ∞.

2. **Sufficient conditions.** To show that under certain slight restrictions the curves (6) actually do give a minimum value to the integral (1), let us consider the difference between the value of the integral for a comparison curve $y = y(x) + \omega(x)$ and its value for the extremal. If we denote this difference by ΔI, it follows by means of Taylor's theorem, that

$$\Delta I = \int_{x_0}^{x_1} \left\{ \frac{2my'(1+y'^2)^{m-1}}{y''}\omega' - \frac{(1+y'^2)^m}{y''^2}\omega'' \right\} dx$$

$$+ \int_{x_0}^{x_1} \left\{ \left[\frac{2m(m-1)\bar{y}'^2(1+\bar{y}'^2)^{m-2}}{\bar{y}''} + \frac{m(1+\bar{y}'^2)^{m-1}}{\bar{y}''} \right] \omega'^2 \right.$$

$$\left. - \frac{2m\bar{y}'(1+\bar{y}'^2)^{m-1}}{\bar{y}''^2}\omega'\omega'' + \frac{(1+\bar{y}'^2)^m}{\bar{y}''^3}\omega''^2 \right\} dx,$$

where $\bar{y}' = y' + \theta\omega'$, $\bar{y}'' = y'' + \theta\omega''$ $(0 < \theta < 1)$.

The first integral, however, vanishes, since the functions y' and y'' refer to the extremals. (Compare equation (2).) Thus, after a simple reduction of the second integral, we have

$$\Delta I = \int_{x_0}^{x_1} \frac{(1+\bar{y}'^2)^m}{\bar{y}''} \left\{ \left[\frac{m\bar{y}'\omega'}{1+\bar{y}'^2} - \frac{\omega''}{\bar{y}''} \right]^2 + \left[\frac{(m-1)\bar{y}'\omega'}{1+\bar{y}'^2} \right]^2 \right.$$

$$\left. + \frac{\omega'^2}{(1+\bar{y}'^2)^2} + \frac{(m-1)\omega'^2}{1+\bar{y}'^2} \right\} dx.$$

If we assume that our extremals are concave upward, that is $y'' > 0$, and consider only comparison curves which are concave upward, we see that ΔI is positive for $m \geqq 1$. If $m = 0$, the sum of the last three terms in the integrand is zero; consequently ΔI is also positive if $m = 0$. Therefore:

If $m \geqq 1$ or $m = 0$ the curves (6) *will give a smaller value to the integral* (1) *than any neighboring curves for which y'' is positive.*

We can also reduce ΔI to the form

$$\Delta I = \tfrac{1}{2} \int_{x_0}^{x_1} \frac{(1+\bar{y}'^2)^m}{\bar{y}''} \left\{ \left[\frac{2m\bar{y}'\omega'}{1+\bar{y}'^2} - \frac{\omega''}{\bar{y}''} \right]^2 + \frac{2m(1-\bar{y}'^2)}{(1+\bar{y}'^2)^2}\omega'^2 + \frac{\omega''^2}{\bar{y}''^2} \right\} dx.$$

This is positive if $m(1-\bar{y}'^2) \geqq 0$. But $m(1-\bar{y}'^2)$ will be positive if $m > 0$ and $|\bar{y}'| < 1$, or if $m < 0$ and $|\bar{y}'| > 1$. Thus we can add to the above conclusion the following:

If $0 < m < 1$, *and we consider only curves having slopes not greater than one in absolute value, the portions of the curves* (6) *for which* $|p| \leqq 1$ *will give a minimum value to the integral* (1).

If $m < 0$, *and we consider only curves having slopes not less than one in absolute value, the portions of the curves* (6) *for which* $|p| \geqq 1$ *will give a minimum value to the integral* (1).

3. **The arbitrary constants and the minimum value of the integral.** If we make use of the conditions that the minimizing curve passes through the point P_0 with the slope p_0 and through the point P_1 with the slope p_1, we obtain from (6) the following four linear equations to solve for the arbitrary constants A, B, C, D:

$$
(7) \qquad \begin{cases} \beta_0 A + \alpha_0 B + C = x_0, \\ \gamma_0 A + \beta_0 B + D = y_0, \\ \beta_1 A + \alpha_1 B + C = x_1, \\ \gamma_1 A + \beta_1 B + D = y_1, \end{cases}
$$

where

$$
(8) \qquad \alpha_i = \int_0^{p_i} \frac{dp}{(1+p^2)^m}, \qquad \beta^i = \int_0^{p_i} \frac{p\,dp}{(1+p^2)^m}, \qquad \gamma_i = \int_0^{p_i} \frac{p^2\,dp}{(1+p^2)^m}.
$$

The determinant of the coefficients of equations (7) has the value

$$
\Delta = \left[\int_{p_0}^{p_1} \frac{p\,dp}{(1+p^2)^m}\right]^2 - \int_{p_0}^{p_1} \frac{dp}{(1+p^2)^m} \cdot \int_{p_0}^{p_1} \frac{p^2\,dp}{(1+p^2)^m},
$$

which, by Schwarz's inequality, is negative. Therefore, *we can always solve for the arbitrary constants* A, B, C, D, *and they are uniquely determined in terms of* $x_0, y_0, p_0, x_1, y_1, p_1$.

Since we wish to use the values of A and B in the next section, we solve for these values here, obtaining

$$
(9) \qquad \begin{cases} A = \dfrac{1}{\Delta}\left[(x_1 - x_0)\displaystyle\int_{p_0}^{p_1} \frac{p\,dp}{(1+p^2)^m} - (y_1 - y_0)\int_{p_0}^{p_1} \frac{dp}{(1+p^2)^m}\right], \\[2ex] B = \dfrac{1}{\Delta}\left[(y_1 - y_0)\displaystyle\int_{p_0}^{p_1} \frac{p\,dp}{(1+p^2)^m} - (x_1 - x_0)\int_{p^0}^{p_1} \frac{p^2\,dp}{(1+p^2)^m}\right]. \end{cases}
$$

By using equations (6) in (1), we discover that the minimum value of the integral I is

(10) $$I = A(y_1 - y_0) + B(x_1 - x_0).$$

4. **Variable slope at an end point.** Let us now consider the case in which the slope at one end point, say P_1, is not fixed. In such a case the value of η' in equation (3) is not zero for $x = x_1$, and therefore, if $I'(0)$ is to vanish, *we must have*

$$\frac{(1+y'^2)^m}{y''^2} = 0$$

at P_1; that is, *there must be a cusp at this point.* We readily find from (5) that *the curve must have at the point* P_1 *the slope* $-B/A$.

5. **End point variable on a vertical line.** Let us now suppose that the slope of the curve at the point P_1 has a fixed finite value, but that the point itself is allowed to move along the line $x = x_1$. To find the conditions that will give the minimum value to the integral I we set $\partial I/\partial y_1 = 0$. Thus from (10) and (9) we get

$$\frac{\partial I}{\partial y_1} = A + (y_1 - y_0)\frac{\partial A}{\partial y_1} + (x_1 - x_0)\frac{\partial B}{\partial y_1} = \frac{2A}{\Delta} = 0.$$

Since Δ can not be infinite if p_0 and p_1 are finite, A *must be equal to zero.*

From the first equation of (9) we find that the value of y_1 which will make $A = 0$ is

$$y_1 = y_0 + \frac{\beta_1 - \beta_0}{\alpha_1 - \alpha_0}(x_1 - x_0),$$

where the α's and β's are defined by (8).

That this value of y_1 does furnish a minimum value for the integral I in the case under consideration is seen by observing that the second derivative,

$$\frac{\partial^2 I}{\partial y_1^2} = -\frac{2}{\Delta}\int_{p_0}^{p_1}\frac{dp}{(1+p^2)^m},$$

is positive, since $\Delta < 0$.

When $A = 0$, we obtain from (7) or (9), $B = (x_1 - x_0)/(\alpha_1 - \alpha_0)$, and thus I reduces to

$$I = B(x_1 - x_0) = \frac{(x_1 - x_0)^2}{\displaystyle\int_{p_0}^{p_1}\frac{dp}{(1+p^2)^m}}.$$

The fact that A must be zero if the end point is variable on a vertical line can also be shown as follows:

The expression for I can easily be changed to the form

$$I = \int_{p_0}^{p_1} (1+p^2)^m \left(\frac{dx}{dp}\right)^2 dp.$$

In this integral the limits are constant, since the curve has a fixed slope p_0 at the point P_0 and crosses the line $x = x_1$ with a fixed slope p_1. Moreover, the values of x corresponding to p_0 and p_1 are constant. Therefore we can obtain, by the usual process of the calculus of variations, for the differential equation to be satisfied by the minimizing curve,

$$(1+p^2)^m \frac{dx}{dp} = \text{constant}.$$

If we designate this constant by B and integrate to obtain x, we get

$$x = \int_0^p \frac{B\,dp}{(1+p^2)^m} + C.$$

Since $dy = p\,dx$, the value of y is the same as in (6) with A set equal to zero.

6. **End point variable on a horizontal line.** If the point P_1 is allowed to vary along the line $y = y_1$, but intersects that line with a fixed slope p_1, we proceed as in the preceding section. We find that

$$\frac{\partial I}{\partial x_1} = (y_1 - y_0)\frac{\partial A}{\partial x_1} + (x_1 - x_0)\frac{\partial B}{\partial x_1} + B = \frac{2B}{\Delta} = 0,$$

and therefore B *must vanish.*

From (9) we find that the value of x_1 which will cause B to vanish is

$$x_1 = x_0 + \frac{\beta_1 - \beta_0}{\gamma_1 - \gamma_0}(y_1 - y_0).$$

That this value of x_1 does give a minimum value to I is shown by the fact that

$$\frac{\partial^2 I}{\partial x_1^2} = -\frac{2}{\Delta}\int_{p_0}^{p_1} \frac{p^2\,dp}{(1+p^2)^m} > 0.$$

If $B = 0$ we see that $A = (y_1 - y_0)/(\gamma_1 - \gamma_0)$, and that

$$I = A(y_1 - y_0) = \frac{(y_1 - y_0)^2}{\displaystyle\int_{p_0}^{p_1} \frac{p^2\,dp}{(1+p^2)^m}}.$$

7. The curve of minimum mean radius of curvature. As an illustration, let us examine the curve whose mean radius of curvature with respect to x is a minimum. If we set up the integral which expresses the mean radius of curvature, we obtain, as was noted in the introduction, the special case $m = \frac{3}{2}$ in (1).

It follows from (6) that the equations of the extremals are

$$x = \int \frac{Ap + B}{(1+p^2)^{\frac{3}{2}}}\,dp, \qquad y = \int \frac{Ap^2 + Bp}{(1+p^2)^{\frac{3}{2}}}\,dp.$$

To integrate, it is convenient to set $p = \tan\tau$. We get

$$(10) \qquad \begin{cases} x = -A\cos\tau + B\sin\tau + C, \\ y = A\log(\sec\tau + \tan\tau) - A\sin\tau - B\cos\tau + D. \end{cases}$$

If $A = 0$, equations (10) reduce to $x - C = B\sin\tau$, $y - D = -B\cos\tau$, a circle.

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